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The Capacity of Matched RS Codes is Zero Over the AWGN Channel

Notations:

block code:	$\mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_M\} \subset \mathcal{A}^n$
codewords of length n:	$\vec{c}_m = (c_{m,1}, \ldots, c_{m,n})$
alphabet:	$\mathcal{A}, \quad (c_{m,i} \in \mathcal{A}), \quad \lvert\mathcal{A}\rvert = q$
code rate:	$R = \frac{\mathrm{ld}\ M}{n} \left[\frac{\text{bit}}{\text{channel use}}\right]$
cardinality:	$\lvert\mathcal{C}\rvert = M = 2^{Rn}$

Linear block codes over GF(q)

Advantages:

- encoding with linear complexity (in n)
- capacity R_c and error exponent $E(R)$ are equal to the capacity and error exponent, resp., of the corresponding coding channel

Disadvantage:

- optimal decoding NP-complete

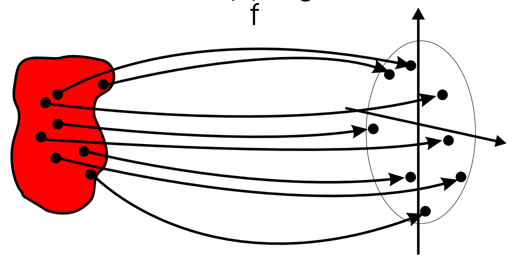
Linear block codes over GF(q)

Specific properties:

- Hamming distance invariant
- Hamming weight characterizable

But: Hamming distance is not detailed enough

Signal point constellation



signal point constellation: $\mathcal{S}_\nu = \{\vec{s}_1, \ldots, \vec{s}_q\} \subset \mathbb{R}^\nu$

mapping into signal space: $f : \mathcal{A} \rightarrow \mathcal{S}_\nu$

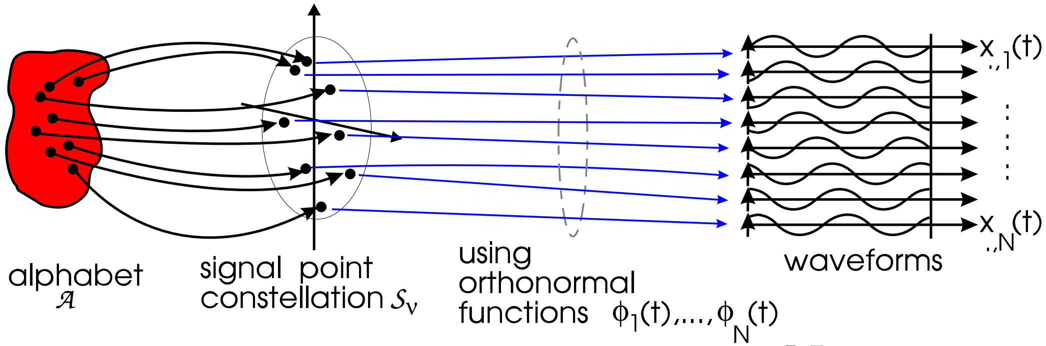
expected signal energy: $E(\mathcal{S}_\nu) = \sum_{l=1}^{q} P[\vec{s}_l] E(\vec{s}_l)$

$E(\vec{s}_l) = |\vec{s}_l|^2$

Euclidean distance distribution:

$$\mathcal{D}(\mathcal{S}_\nu) = \{d_E(\vec{s}_l, \vec{s}_j) \mid l < j; l, j = 1, \ldots, q\}$$

Waveform channel



orthonormal functions: $\Phi = \{\phi_i(t)\}_{i=1}^{N}$

coded waveforms:

$$x_m(t) = \sum_{i=1}^{N} x_{m,i} \cdot \phi_i(t); \quad m = 1, \ldots, M$$

mean-square difference between two waveforms:

$$\delta_{m,j}^2 = \int_0^T [x_m(t) - x_j(t)]^2 dt = d_E^2(\vec{x}_m, \vec{x}_j)$$

For waveform channels a relevant distance measure is

Euclidean distance

Euclidean distance

The Euclidean distance precisely expresses the quantity of the difference

- between signal points,
- between Euclidean representations $\mathcal{E}$ of codewords from $\mathcal{C}$ and
- between corresponding coded waveforms.

But: distance invariance and weight characterization are <u>no</u> longer valid.

Matched mappings

Search for mappings $f : \mathcal{A} \rightarrow \mathcal{S}_\nu$ preserving the distance invariance and weight characterization for induced Euclidean distances.
Condition for f in case $\mathcal{A}$ is provided with a group structure $(G, *)$ [Löliger]: $\forall g, g' \in G$

$$d_E(f(g), f(g')) = d_E(f(g^{-1} * g'), f(e))$$

The corresponding vector constellation $\mathcal{S}_\nu^*$ is Euclidean distance invariant and Euclidean weight representable.

RS codes and matched mappings

Some RS codes are bad on the AWGN channel for matched mappings defined on the modulo q addition of the elements of GF(q).

Proof (8 Parts):

1. Any vector constellation $\mathcal{S}_\nu^*$ matched to a group is a group code for the Gaussian channel (see Löliger 1992).

2. Group codes for the Gaussian channel are Euclidean distance invariant spherical codes (see Slepian 1968).

3. For RS codes $q = n + 1$ holds. Thus, for $n \to \infty$, constant $\nu \in \mathbb{N}$ and constant average energy, the minimum Euclidean distance in $\mathcal{S}_\nu^*$ tends to zero.

4. Since Euclidean representations $\mathcal{E}^*$ of block codes $\mathcal{C}$ are Euclidean distance invariant [Löliger 91], $\mathcal{E}^*_{\mathrm{RS}}$ are also Euclidean distance invariant.

5. SNR on the AWGN channel is constant for $\mathcal{E}^*$, when $n \to \infty$ and ν and $E(\mathcal{S}^*_\nu)$ remain constant, because

$$\mathrm{SNR} = \frac{nE}{n\nu\sigma^2} = \frac{E}{\nu\sigma^2} = \mathrm{const}$$

6. RS codes with generator polynomials not divisible by $(x-1)$ contain all codewords of the form $\vec{x}^{(l)} = (\psi_l, \psi_l, \ldots, \psi_l)$, $\psi_l \in \mathrm{GF}(q)$, $l = 1, \ldots, q$.

7. Let $f(\psi_l)$ and $f(\psi_j)$ be of minimum Euclidean distance ϵ. There exist two codewords of the form $\vec{x}_{\mathcal{E}}^{(l)} = (f(\psi_l), f(\psi_l), \ldots, f(\psi_l))$ and $\vec{x}_{\mathcal{E}}^{(j)} = (f(\psi_j), f(\psi_j), \ldots, f(\psi_j))$ such that the normalized Euclidean distance $\underline{d}_E^0 = \epsilon$ tends to zero for $n \to \infty$ (see part 3).

8. For constant SNR the error exponent of constant rate distance invariant block code families is zero (see [Lazic,Senk 92]), and thus also the capacity, when the normalized minimum distance tends to zero as $n \to \infty$.

Conclusion

The items 1. to 8. imply that the family of matched Euclidean representations of Reed-Solomon codes $\mathcal{E}^*_{\mathrm{RS}}$, whose generator polynomials do not contain the factor $(x-1)$, have a *family error exponent*, and thus a *family capacity* for the AWGN channel, equal to zero.